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Partner Heterogeneity Increases the Set of Strict Nash Networks*

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Abstract

Galeotti et al. (2006, [2]) show that all minimal networks can be strict Nash in two-way flow models with full parameter heterogeneity while only inward pointing stars and the empty network can be strict Nash in the homogeneous parameter model of Bala and Goyal (2000, [1]). In this note we show that the introduction of partner heterogeneity plays a major role in substantially increasing the set of strict Nash equilibria.

JEL Classification: C72, D85

Key Words: Nash networks, two-way flow models, point contrabasis.

1 Introduction

In their seminal paper on Nash networks with homogeneous players, Bala and Goyal (2000, [1]) find that in the two-way flow model the equilibrium set is very small. In particular they show that only the empty network and the inward pointing star (a star network where the central agent forms all the links) can be strict Nash networks. Galeotti et al. (2006, [2]) introduce heterogeneity in this model by allowing costs and benefits of links to depend on the identity of the player who is forming the links and acquiring information. Hence player i always gets the same information V_i from the other players and pays a cost c_i for all her links. Under this type of heterogeneity, which we call *player heterogeneity*, they find no change in the set of strict Nash networks. Subsequently, they introduce full heterogeneity in values by making them dependent on the link, *i.e.*, player i gets value V_{ij} from player j, and find that strict Nash networks can have components, but each such component is still an inward pointing star. When they additionally introduce full cost heterogeneity, *i.e.*, allow costs to depend on the link as well (denoted by c_{ij}), they find that the set of strict Nash networks increases dramatically. All strict Nash networks are now minimal networks, that is networks with components where the deletion of any link increases the number of components.

In this paper, we ask whether the introduction of full heterogeneity of players is the only way to destabilize the Bala and Goyal's results. To answer this question, we allow heterogeneity to be *partner dependent*, that is the value acquired from player j is V_j for all players, and the costs of linking to player i's partner player j are c_j. In practice, such situations where accessing different agents involves different costs and yields different benefits of course arise quite frequently. We show that this type of heterogeneity substantially increases the set of equilibrium networks and thus is the driving force behind the increase in the size of strict Nash networks in models with heterogeneity. In other words, it is not necessary to have two degrees of freedom in the cost parameter c_{ij} to destabilize the results of the homogeneous case. One degree of freedom in values and costs, provided it is partner and not player dependent, can lead to a larger set of strict Nash networks.

2 Model setup

Let $N = \{1, \ldots, n\}$ be the set of players. Each player i chooses a strategy $\boldsymbol{g}_i = (\boldsymbol{g}_{i,1}, \ldots, \boldsymbol{g}_{i,i-1}, \boldsymbol{g}_{i,i+1}, \boldsymbol{g}_{i,n})$ where $\boldsymbol{g}_{i,j} \in \{0, 1\}$ for all $j \in N \setminus \{i\}$. The interpretation of $\boldsymbol{g}_{i,j} = 1$ is that player i forms a link with player $j \neq i$, and the interpretation of $\boldsymbol{g}_{i,j} = 0$ is that i forms no link with player j. By convention, we assume that player i cannot form a link with herself. In the following we only use pure strategies. Let $\mathcal{G}_i$ be the set of all strategies of player $i \in N$. Network relations among players are formally represented by directed networks whose nodes are identified with the players.

A network $\boldsymbol{g} = (N, E)$ is a pair of sets: the set N of players and the set $E \subset N \times N$ of directed links. In the following, we denote by $E(\boldsymbol{g})$ the set of links of network $\boldsymbol{g}$. A link from j to i is denoted by j, i. A network $\boldsymbol{g}$ is transitive if $i, j \in E(\boldsymbol{g})$ whenever both $i, k \in E(\boldsymbol{g})$ and $k, j \in E(\boldsymbol{g})$. A network $\boldsymbol{g}$ which satisfies a property, say p_1, is minimal if there does not exist a network $\boldsymbol{g}'$ such that $E(\boldsymbol{g}') \subsetneq E(\boldsymbol{g})$ and $\boldsymbol{g}'$ satisfies p_1. The transitive closure of a network $\boldsymbol{g}$ is the minimal transitive network containing $\boldsymbol{g}$. A network $\boldsymbol{g}$ is symmetric if $i, j \in E(\boldsymbol{g})$ implies $j, i \in E(\boldsymbol{g})$. The symmetrized network associated with network $\boldsymbol{g}$ is the minimal symmetric network which contains $\boldsymbol{g}$. Let $\mathcal{G}$ be the set of all simple directed networks. Let $f : \mathcal{G} \to \mathcal{G}$, $\boldsymbol{g} \mapsto f(\boldsymbol{g})$ be a mapping which associates with network $\boldsymbol{g}$ the transitive closure of $\boldsymbol{g}$. Let $h : \mathcal{G} \to \mathcal{G}$, $\boldsymbol{g} \mapsto h(\boldsymbol{g})$ be a mapping which associates with network $\boldsymbol{g}$ the symmetrized network of $\boldsymbol{g}$. Let $\dot{\boldsymbol{g}} = f(\boldsymbol{g})$,

$\bar{\boldsymbol{g}} = h(\boldsymbol{g})$, and the composition be $\hat{\boldsymbol{g}} = f \circ h(\boldsymbol{g})$.

Define $N_i(\boldsymbol{g}) = \{i \text{ and } j \in N \setminus \{i\} \mid \hat{g}_{i,j} = 1\}$ as the set of players who are observed by player i with the convention that player i always "observes" himself. We assume that values and costs are partner heterogeneous. More precisely, each player i obtains $V_j > 0$ from each player $j \in N_i(\boldsymbol{g}) \setminus \{i\}$, and incurs a cost $c_j > 0$ when she forms a link with player $j \neq i$. Also since we wish to focus only on the network formed, we assume that player i obtains resources of other players $j \neq i$, but no additional resources from herself. Indeed, player i obtains her own resources even if she forms no links and there is no network.

Let $\phi : \mathbb{R}^2_+ \to \mathbb{R}$, $\phi : (x, y) \mapsto \phi(x, y)$, be such that $\phi(x, y)$ is strictly increasing in x and strictly decreasing in y. Define each agent's payoff function $\pi_i(\boldsymbol{g})$, as

$$\pi_i(\boldsymbol{g}) = \phi\left(\sum_{j \in N_i(\boldsymbol{g}) \setminus \{i\}} V_j, \sum_{j \in N} \boldsymbol{g}_{i,j} c_j\right). \tag{1}$$

Given the properties we have assumed for the function ϕ, the first term can be interpreted as the "benefits" that agent i receives from her links, while $\sum_{j \in N \setminus \{i\}} g_{i,j} c_j$ measures the "costs" associated with forming them. A special case of (1) is when payoffs are linear:

$$\pi_i^L(\boldsymbol{g}) = \sum_{j \in N_i(\boldsymbol{g}) \setminus \{i\}} V_j - \sum_{j \in N} \boldsymbol{g}_{i,j} c_j \tag{2}$$

We now introduce the set of players who have the minimal cost of setting links, $S_{i_0} = \{j \in N \mid j \in \arg\min_{j \in N}\{c_j\}\}$, with i_0 as a typical member of the set S_{i_0}. Let s_{i_0} be the cardinality of the set S_{i_0}. Finally, let $\mathcal{H} = \{j \in N \mid c_j < V_j\}$ be the set of players whose value is greater than the cost of linking to them.

Network Definitions. We now provide some network definitions that are used in this note.[1] For a directed graph, $\boldsymbol{g} \in \mathcal{G}$, a *path* $P_{j,i}(\boldsymbol{g})$ of length m in $\boldsymbol{g}$ from players j

[1]Almost all these definitions come from Harary, Norman and Cartwright (1965, [3]), sometimes with

to player i, $i \neq j$, is a finite sequence $i_0, i_1, \ldots, i_m$ of distinct players such that $i_0 = i$, $i_m = j$ and $\boldsymbol{g}_{i_k,i_{k+1}} = 1$ for $k = 0, \ldots, m-1$. A *chain* $C_{i,j}(\boldsymbol{g})$ in $\boldsymbol{g}$ between player j and player i, $i \neq j$, is a finite sequence $i_0, i_1, \ldots, i_m$ of distinct players such that $i_0 = i$, $i_m = j$ and $\max\{\boldsymbol{g}_{i_k,i_{k+1}}, \boldsymbol{g}_{i_{k+1},i_k}\} = 1$ for $k = 0, \ldots, m-1$. Let us denote by $\mathcal{C}_{i,j}(\boldsymbol{g})$ the set of all chains between i and j in $\boldsymbol{g}$. The length of a chain $C_{i,j}(\boldsymbol{g})$ between i and j in $\boldsymbol{g}$ is denoted by $\ell(C_{i,j}(\boldsymbol{g}))$, and consists in the number of links between player i and j in $C_{i,j}(\boldsymbol{g})$. Let $d_{\boldsymbol{g}}(i,j) = \operatorname{argmin}_{C_{i,j}(\boldsymbol{g}) \in \mathcal{C}_{i,j}}\{\ell(C_{i,j}(\boldsymbol{g}))\}$ be the geodesic distance between i and j in $\boldsymbol{g}$. The diameter of a network $\boldsymbol{g}$ is defined as $\max_{i \in N, j \in N}\{d_{\boldsymbol{g}}(i,j)\}$. A network is connected if $\hat{\boldsymbol{g}}_{i,j} = 1$ for all $i \in N$ and $j \in N \setminus \{i\}$. Let us denote by $\mathcal{G}^{\mathrm{mc}}$ the set of minimally connected networks. Given a network $\boldsymbol{g}$, we define a *component* as a network $\boldsymbol{g}'$ such that $E(\boldsymbol{g}') \subset E(\boldsymbol{g})$ and $E(\boldsymbol{g}') \subset D \times D$ with $D \subset N$ such that for all players $i \in D$ and $j \in D \setminus \{i\}$ we have $\hat{\boldsymbol{g}}_{i,j} = 1$, and for all $i \in D$ and $j \notin D$ we have $\hat{\boldsymbol{g}}_{i,j} = 0$.

Various kinds of architectures play a role in this note. The *empty network* $\boldsymbol{g}^e$, is a network such that for all $i \in N, j \in N \setminus \{i\}$, we have $\boldsymbol{g}^e_{i,j} = 0$. A network $\boldsymbol{g}$ is a *star* if there is a player i such that $\bar{\boldsymbol{g}}_{i,j} = 1$ for all $j \in N \setminus \{i\}$ and $\bar{\boldsymbol{g}}_{\ell,j} = 0$ for all $\ell \in N \setminus \{i\}$ and $j \in N \setminus \{i, \ell\}$. The network $\boldsymbol{g}$ is an *inward pointing star* if it is a star and for the center player i, we have $\boldsymbol{g}_{i,j} = 1$ for all $j \in N \setminus \{i\}$.

Let $Q_i(\boldsymbol{g}) = \{i \text{ and } j \in N \setminus \{i\} \mid \dot{\boldsymbol{g}}_{i,j} = 1\}$ be the set of players which contains player i and players $j \in N \setminus \{i\}$ such that there exists a path from player j to player i. This set can be extended to a set of points $N' \subset N$: $Q_{N'}(\boldsymbol{g}) = \{i \in N' \text{ and } j \in N \setminus N' \mid \exists (i,j) \in N' \times N \setminus N', \dot{\boldsymbol{g}}_{i,j} = 1\}$. A *point contrabasis* of a network $\boldsymbol{g}$, $B(\boldsymbol{g})$, is a minimal set (for the inclusion relation $\subset$) of players such that $Q_{B(\boldsymbol{g})} = N$. An i-point contrabasis, $B_i(\boldsymbol{g})$, is a point contrabasis of $\boldsymbol{g}$ such that all players $j \in B_i(\boldsymbol{g})$ satisfy $\boldsymbol{g}_{j,i} = 1$. Let $I_i(\boldsymbol{g}) = \{j \in N \mid \boldsymbol{g}_{j,i} = 1\}$ be the set of players who form a link with player i in $\boldsymbol{g}$.

slight modifications.

Recall that i_0 is a typical member of S_{i_0}. A B_{i_0}-network is a network which satisfies the following properties: $|I_{i_0}(\boldsymbol{g})| \geq 2$, $|I_j(\boldsymbol{g})| < 2$ for all $j \neq i_0$, and $I_{i_0}(\boldsymbol{g}) = B_{i_0}(\boldsymbol{g})$. In such a network, there is only one player (the minimal cost player) with whom at least two players have formed a link, and this forms a point contrabasis of the network. We know from Harary, Norman, Cartwright that every network has a point contrabasis (see Corollary, [3] 4.2a', p.93). A network $\boldsymbol{g}$ is a *con-tree* (converse tree) if it is minimally connected and contains a unique player $i \in N$ such that for all players $j \in N \setminus \{i\}$, there is a path from j to i in $\boldsymbol{g}$. It is clear that the point contrabasis of a con-tree is a singleton.

Nash Networks and Strict Nash Networks. Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network obtained when all of player i's links are removed. The network $\boldsymbol{g}$ can be written as $\boldsymbol{g} = \boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}$ where '$\oplus$' indicates that $\boldsymbol{g}$ is formed as the union of the links of $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$. Likewise if we write $\boldsymbol{g} = \boldsymbol{g}' \ominus \boldsymbol{g}''$, then $\boldsymbol{g}$ is the difference between the links of $\boldsymbol{g}'$ and $\boldsymbol{g}''$. The strategy $\boldsymbol{g}_i$ is said to be a best response of player i to the network $\boldsymbol{g}_{-i}$ if:

$$\pi_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq \pi_i(\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i}), \text{ for all } \boldsymbol{g}'_i \in \{0,1\}^{n-1}.$$

The set of all of player i's best responses to $\boldsymbol{g}_{-i}$ is denoted by $\mathcal{BR}_i(\boldsymbol{g}_{-i})$. A network $\boldsymbol{g}$ is said to be a Nash network if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each player $i \in N$. We define a strict best response and a strict Nash network by replacing '$\geq$' with '$>$'.

3 Characterization of strict Nash networks

We start with a preliminary result that is stated without proof. The proof is similar to one found in many network papers (see for instance Galeotti et al. [2]).

Lemma 1 *Suppose the payoff function satisfies equation (1). A non empty strict Nash network* $\boldsymbol{g}$ *is minimally connected.*

We know from Bala and Goyal (2000, [1], Proposition 4.2) that a non empty strict Nash network $\boldsymbol{g}$ in the homogeneous case ($V_j = V$ and $c_j = c$, for all $j \in N$) is such that $|B(\boldsymbol{g})| = 1$. Also from Galeotti et al. (adaptation of proposition 3.1, 2006, [2]), we know that in the player heterogeneity model with linear payoffs, a non empty strict Nash network $\boldsymbol{g}$ is such that $|B(\boldsymbol{g})| = 1$. Indeed, in both cases, the inward pointing star is the unique non empty strict Nash network. However, the introduction of partner heterogeneity dramatically increases the set of strict Nash networks. In the following proposition we characterize the architecture of these networks. To establish our first proposition, we recall a result of digraph theory.

Theorem 1 ***(Harary, Norman, Cartwright** (1965, [3], Theorem 4.3', p.93) Every minimally connected network has a unique point contrabasis.*

Proposition 1 *Suppose the payoff function satisfies equation (1). A non empty strict Nash network* $\boldsymbol{g}$ *is either a minimally connected* B_{i_0}*-network or a con-tree.*
Suppose the payoff function satisfies equation (2). There exist parameters $(c_j)_{j\in N}$*,* $(V_j)_{j\in N}$*, such that any minimally connected* B_{i_0}*-network and any con-tree are strict Nash networks.*

Proof See Appendix. □

Proposition 1 extends to the case where the payoff function of each player i satisfies

$$\pi_i^V(\boldsymbol{g}) = \sum_{j\in N_i(\boldsymbol{g})\setminus\{i\}} V_{ij} - \sum_{j\in N} \boldsymbol{g}_{i,j} c_j.$$

This payoff function captures situations where values are fully heterogeneous while costs are partner heterogeneous. Using the same arguments as in the proof of Proposition 1 it can be shown that equilibrium networks are not always connected: they contain some components. Moreover, if we denote by i^D the minimal cost player of the component D, then components of equilibrium networks are either B_{i^D}-networks, or con-trees.

We now examine the relationship between the set of strict Nash networks in the homogeneous model and the partner heterogeneity model. For the sake of exposition, we focus on the linear case.

Remark 1 *Homogeneous model (Bala and Goyal (2000, [1]) vs Partner Heterogeneity model.* Recall that inward pointing stars are the unique strict Nash networks in the linear homogeneous model, when $V > c$. In the partner heterogeneity linear model, it is always possible to form a network of diameter strictly greater than 2 if $\mathcal{H} = N$, and there is at least one player who has a link cost different from the others. This is easy to show.

Let there be only one player, say i, whose cost is different from the cost of other players. Then, there are two possibilities: either player i is the lowest cost player, or i is the highest cost player. First, let i be the lowest cost player. Then, for a network $\boldsymbol{g}$ if there exist two players, $j \in N \setminus \{i\}$ and $k \in N \setminus \{i, j\}$, such that $\boldsymbol{g}_{j,\ell} = 1$ for all $\ell \in N \setminus \{j, k\}$, $\boldsymbol{g}_{i,k} = 1$, and there exist no other links, then $\boldsymbol{g}$ is a strict Nash network. Second, let player i be the highest cost player. If there exist two players $j \neq i$ and $k \neq i$ in $\boldsymbol{g}$ such that $\boldsymbol{g}_{j,\ell} = 1$ for all $\ell \in N \setminus \{i, j\}$, $\boldsymbol{g}_{k,i} = 1$, and there exist no other links, then $\boldsymbol{g}$ is a strict Nash network.

Thus, the result of the homogeneous case *cannot* be found in the partner heterogeneity model. This sharply contrasts with what we find in the player heterogeneity model. This is true even if only one agent has costs different from others and this difference is very low. Consequently, it is not possible to use continuity arguments to study the set of strict Nash networks of partner heterogeneity model from the set of strict Nash networks of homogeneous model.

Let us denote by $\bar{\mathcal{G}}^{\text{mc}}$ the set of symmetrized minimally connected networks and by $\mathcal{G}^{\text{ct}}$ the set of networks which are con-trees. We now present a result which allows us to study the relationship between the set of strict Nash networks according to the type of heterogeneity.

Proposition 2 *Suppose the payoff function satisfies equation (1). We have* $h(\mathcal{G}^{ct}) = \bar{\mathcal{G}}^{mc}$.

Proof It is straightforward that $\mathcal{G}^{\text{mc}} \supset \mathcal{G}^{\text{ct}}$. We now show that $\bar{\mathcal{G}}^{\text{mc}} \subset h(\mathcal{G}^{\text{ct}})$. Let $\boldsymbol{g} \in \bar{\mathcal{G}}$. We will show that $\boldsymbol{g}$ is a symmetrized network of a network $\boldsymbol{g}' \in \mathcal{G}^{\text{ct}}$. Let $N' = \{i \in N \mid i \in \text{argmin}_{j \in N}\{\mid I_j(\boldsymbol{g}) \mid\}\}$. We choose one player, say i', in N'. For all players $i \in N \setminus \{i'\}$ such that $\max\{\boldsymbol{g}_{i',i}, \boldsymbol{g}_{i,i'}\} = 1$, we let $\boldsymbol{g}'_{i',i} = 1$. Likewise for all players $j \in N \setminus \{i'\}$ and $k \in N \setminus \{i', j\}$, who belong to the same chain $C_{i',\ell}(\boldsymbol{g})$, we impose the following rule: If $\boldsymbol{g}_{k,j} = \boldsymbol{g}_{j,k} = 1$ and $d_{\boldsymbol{g}}(i', j) < d_{\boldsymbol{g}}(i', k)$, then we have $\boldsymbol{g}'_{j,k} = 1$ and $\boldsymbol{g}'_{k,j} = 0$. It is clear that with this rule, we obtain from any network $\boldsymbol{g} \in \bar{\mathcal{G}}^{\text{mc}}$ a network $\boldsymbol{g}'$ which is a con-tree. $\square$

Proposition 2 stresses that every minimally symmetrized connected network can emerge from strict Nash networks in the partner heterogeneity model. Using similar arguments, we can show that if the payoff function of each player i is π_i^V, then every minimally symmetrized network can emerge from strict Nash networks in the model. Moreover, we know from Galeotti et al. (2006, [2], Proposition 3.2) that in the model with full heterogeneity a strict Nash network is a minimal network. In the following remark, we study the role of heterogeneity for the set of strict Nash networks. To simplify our analysis, we focus on the linear case.

Remark 2 *Set of potential strict Nash networks: The role of heterogeneity.* From Propositions 1 and 2 in the current paper and Propositions 3.1 and 3.2 due to Galeotti et al. 2006, [2] we can establish the following facts.

1. The set of strict Nash networks in the partner heterogeneity model allows for more architectures than the empty network and the inward pointing stars found in the homogeneous and player heterogeneous models. In other words, even one degree of freedom (V_j,c_j) is sufficient to obtain a set of strict Nash networks which is quite large.

2. To obtain a non connected network as a non empty strict Nash network it is necessary to introduce two degrees of freedom (V_{ij},c_{ij}) in values or costs.

3. By Proposition 2, it is clear that the two degrees of freedom in costs (c_{ij}) only help relax the constraints on the direction of links of potential strict Nash networks. Indeed, the model with payoff function π_i^V, allows for the same set of potential strict Nash architectures as the full heterogeneity model, provided the direction of links is not taken into account. Moreover, potential strict Nash connected networks have very similar properties (when considering player centrality or average distance between players) in the partner dependency model and the full heterogeneity model.

4 Conclusion

We have shown that while the introduction of partner heterogeneity leads to a richer set of strict Nash architectures than player heterogeneity, this class is not quite as large as the equilibrium set under full heterogeneity in values and costs. However, from the point of view of network properties, sets of strict Nash connected architectures found in full heterogeneity model and in partner heterogeneity model are quite similar, and are very different from the set of strict Nash architectures found in player heterogeneity model. In this sense the introduction of the heterogeneity associated with the partner is responsible for the dramatic change in results when we go from the homogeneous or player dependency model to the full heterogeneity model.

Appendix

Proof of proposition 1. First, we show that if the payoff function satisfies equation (1), then a non empty strict Nash network $\boldsymbol{g}$ is either a minimally connected B_{i_0}-network or a con-tree. From Lemma 1, we know that a non empty strict Nash network $\boldsymbol{g}$ is minimally connected, we begin the proof with three additional properties of $\boldsymbol{g}$.

1. If there are two players k and ℓ such that $\boldsymbol{g}_{k,i} = \boldsymbol{g}_{\ell,i} = 1$, then there does not exist player j such that $\max\{\boldsymbol{g}_{j,k}, \boldsymbol{g}_{j,\ell}\} = 1$. Indeed, since $\boldsymbol{g}_{k,i} = \boldsymbol{g}_{\ell,i} = 1$, we have $N_k(\boldsymbol{g}) = N_\ell(\boldsymbol{g})$ and $c_i < \min\{c_\ell, c_k\}$ otherwise either player k or player ℓ is not playing a strict best response. The assertion about j follows from this.

2. If there are three players i, j and k such that $\boldsymbol{g}_{i,j} = \boldsymbol{g}_{k,i} = 1$, then $\boldsymbol{g}_{j',j} = 0$ for all $j' \in N \setminus \{i, j\}$. Again, it is clear that $N_i(\boldsymbol{g}) = N_j(\boldsymbol{g})$ and $c_i < c_j$ otherwise player k has an incentive to form a link with player j to play a strict best response. It follows that if player $j' \in N \setminus \{i, j\}$ forms a link with player j, then she is not playing a strict best response.

3. If there exists a player i, in a strict Nash network $\boldsymbol{g}$, such that $|I_i(\boldsymbol{g})| \geq 2$, then $s_{i_0} = 1$ and player i is i_0. To show this assume that there are two players i and

j such that $|I_i(\boldsymbol{g})| \geq 2$ and $|I_j(\boldsymbol{g})| \geq 2$. Then there are players $k_i \in N$, $k_i' \in N$ and $k_j \in N$, $k_j' \in N$ such that $\boldsymbol{g}_{k_i,i} = \boldsymbol{g}_{k_i',i} = 1$ and $\boldsymbol{g}_{k_j,j} = \boldsymbol{g}_{k_j',j} = 1$. Since $\boldsymbol{g}$ is minimally connected it follows that either $i \in N_j(\boldsymbol{g} \ominus i, k_i)$ or $i \in N_j(\boldsymbol{g} \ominus i, k_i')$. Likewise, we have either $j \in N_i(\boldsymbol{g} \ominus j, k_j)$ or $j \in N_i(\boldsymbol{g} \ominus j, k_j')$. Without loss of generality, we assume that $i \in N_j(\boldsymbol{g} \ominus i, k_i)$ and $j \in N_i(\boldsymbol{g} \ominus j, k_j)$. Clearly if k_i deletes her link with i and forms a link with player j, then she obtains the same resources as in the network $\boldsymbol{g}$. It follows that $c_i < c_j$, otherwise $\boldsymbol{g}$ is not a strict Nash network. Using a similar argument for player k_j it follows that $c_j < c_i$, otherwise $\boldsymbol{g}$ is not a strict Nash network. A contradiction.

Next assume that there is a player $i \notin S_{i_0}$ such that $|I_i(\boldsymbol{g})| \geq 2$, and let $k_i \in N$, $k_i' \in N$ such that $\boldsymbol{g}_{k_i,i} = \boldsymbol{g}_{k_i',i} = 1$. It is obvious that $k_i \notin S_{i_0}$ and $k_i' \notin S_{i_0}$. By minimality of $\boldsymbol{g}$, we have either $i \in N_{i_0}(\boldsymbol{g} \ominus i, k_i)$ or $i \in N_{i_0}(\boldsymbol{g} \ominus i, k_i')$. Without loss of generality, assume that $i \in N_{i_0}(\boldsymbol{g} \ominus i, k_i)$. It follows that if player k_i deletes her link with i and forms a link with i_0, she obtains the same resources as in $\boldsymbol{g}$. Since $c_{i_0} < c_i$, $\boldsymbol{g}$ is not a strict Nash network, a contradiction. Finally, if $s_{i_0} > 1$, then we use the same kind of arguments to show that $\boldsymbol{g}$ is not a strict Nash network.

Suppose that $s_{i_0} > 1$. We know that a non empty strict Nash network $\boldsymbol{g}$ is minimally connected. From points 1, 2 and 3, it is clear that $\boldsymbol{g}$ is a con-tree. Suppose now that $s_{i_0} = 1$. Recall that $B(\boldsymbol{g})$ is the point contrabasis of network $\boldsymbol{g}$. It is straightforward that if $|B(\boldsymbol{g})| = 1$, then strict Nash networks are con-trees. We now show that if $|B(\boldsymbol{g})| > 1$, then strict Nash networks are B_{i_0}-networks. We know from point 3 that at most one player, i_0, is such that $|I_{i_0}(\boldsymbol{g})| \geq 2$ and from Theorem 1 we know that there is a unique point contrabasis in $\boldsymbol{g}$. Assume now that there is a player $i \in B(\boldsymbol{g}) \setminus I_{i_0}(\boldsymbol{g})$. It means that there is no player j such that $\boldsymbol{g}_{j,i} = 1$ otherwise $B(\boldsymbol{g})$ is not minimal. It follows that player i has formed a link with a player j, which allows her to obtain resources of i_0 (otherwise $\boldsymbol{g}$ is not minimally connected). Therefore, if player i replaces her link with j by a link with player i_0 she obtains the same resources and since $\boldsymbol{g}$ is a strict Nash network, we must have $c_j < c_{i_0}$, a contradiction. So, we have $B(\boldsymbol{g}) \subset I_{i_0}(\boldsymbol{g})$. Since $|B(\boldsymbol{g})| > 1$ and $B(\boldsymbol{g}) \subset I_{i_0}(\boldsymbol{g})$, we have $|I_{i_0}(\boldsymbol{g})| > 1$. In that case, we know by Point 1 that there is no player who has formed a link with $\ell \in I_{i_0}(\boldsymbol{g})$. It follows that for all $\ell \in I_{i_0}(\boldsymbol{g})$, we have $\ell \in B(\boldsymbol{g})$ and $B(\boldsymbol{g}) \supset I_{i_0}(\boldsymbol{g})$. As a result, strict Nash networks are B_{i_0}-networks.

Second, we show that if the payoff function satisfies equation (2), then there exist parameters $(c_j)_{j\in N}$, $(V_j)_{j\in N}$, such that any minimally connected B_{i_0}-network and any con-tree are strict Nash networks.

Consider a minimally connected B_{i_0}-network $\boldsymbol{g}$. Denote by $P^{i_0}_{j,i}(\boldsymbol{g})$ a path from player j to player i through player i_0 and denote by $P^{-i_0}_{j,i}(\boldsymbol{g})$ a path from player j to player i which does not contain player i_0. Let $\mathcal{H} = N$. If there is a path from player i_1 to i_m, $P^{-i_0}_{i_1,i_m}(\boldsymbol{g})$, which consists of the following sequence of players: $i_1, i_2, \ldots, i_{m-1}, i_m$ in $\boldsymbol{g}$, then $c_{i_k} > c_{i_{k+1}}$ for all $k \in \{1, \ldots, m-1\}$. Finally, assume that if there is a path $P^{i_0}_{i_1,i_m}(\boldsymbol{g}) = \{i_1, i_2, i_3 \ldots, i_{m-1}, i_0, i_m\}$ in $\boldsymbol{g}$, then $c_{i_k} > c_{k+1}$ for all $k \in \{1, \ldots, m-1\}$. Recall that by definition, in a minimally connected B_{i_0}-network no player forms a link with $j \in B_{i_0}(\boldsymbol{g})$ and only players who belong to $B_{i_0}(\boldsymbol{g})$ form a link with i_0.

Since $\boldsymbol{g}$ is minimally connected and $V_j > c_j$ for all players $j \in N$, no player has an incentive to delete a link with a player $j \in N$ in $\boldsymbol{g}$ if she does not replace this link by another link. We show that no player $j \in N$ can replace a link in $\boldsymbol{g}$ and preserve her payoff. Clearly, if player $i_k \in P^{-i_0}_{i_1,i_m}(\boldsymbol{g})$ chooses to replace the link i_{k-1}, i_k, then in order to access to the same amount of resources as before, she must form a link with a player $i_{k'} \in P^{-i_0}_{i_1,i_m}(\boldsymbol{g})$ with $k' < k-1$, since $N_{i_k}(\boldsymbol{g} \ominus i_{k+1}, i_k \oplus i_{k'}, i_k) = N_k(\boldsymbol{g})$ in a minimally connected B_{i_0}-network. However, since $c_{i_{k'}} > c_{i_{k-1}}$ for all $k' < k-1$, player k has no incentive to replace one of her links. Likewise, if $i_k \in P^{i_0}_{i_1,i_m}(\boldsymbol{g})$ and $i_k \notin B_{i_0}(\boldsymbol{g})$, then we use the same argument as before to show that player i_k has no incentive to delete or to replace one of her links. Finally, if $i_k \in P^{i_0}_{i_1,i_m}(\boldsymbol{g})$ and $i_k \in B_{i_0}(\boldsymbol{g})$, then player i_k has no incentive to replace her link with i_0, since $c_{i_0} < c_j$ for all $j \in N \setminus \{i_0\}$.

By using similar arguments, it is obvious that there exist $(c_j)_{j\in N}$ and $(V_j)_{j\in N}$ such that any con-tree can be a strict Nash network. $\square$

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